ON THE J-ANTI-INVARIANT COHOMOLOGY OF ALMOST COMPLEX 4-MANIFOLDS

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Abstract. For a compact almost complex 4-manifold (M, J), we study the subgroups $H_J^{\pm}$ of $H^2(M, \mathbb{R})$ consisting of cohomology classes representable by J-invariant, respectively, J-anti-invariant 2-forms. If $b^+ = 1$, we show that for generic almost complex structures on M, the subgroup H_J^- is trivial. Computations of the subgroups and their dimensions $h_J^{\pm}$ are obtained for almost complex structures related to integrable ones. We also prove semi-continuity properties for $h_J^{\pm}$.

1. Introduction

For any almost complex manifold (M, J), the last two authors [24] introduced certain subgroups of the de Rham cohomology groups, naturally defined by the almost complex structure. These subgroups are interesting almost complex invariants and there are several works already devoted to their study [12], [1], [11], [2]. Particularly important are the subgroups H_J^+, H_J^- of $H^2(M, \mathbb{R})$, defined as the sets of cohomology classes which can be represented by J-invariant, respectively, J-anti-invariant real $2-$forms. They naturally appear in the relationship between the compatible symplectic cone and the tamed symplectic cone of a given compact almost complex manifold [24]. All of the above quoted works consider the problem of whether or not the subgroups H_J^+, H_J^- induce a direct sum decomposition of $H^2(M, \mathbb{R})$. This is known to be true for integrable almost complex structures J which admit compatible Kähler metrics on compact manifolds of any dimension. In this case, the induced decomposition is nothing but the classical (real) Hodge-Dolbeault decomposition of $H^2(M, \mathbb{R})$ (see [24], [12], [11]). On the other hand, examples from [12], [1], [2] show that there exist almost complex structures, even integrable ones, on compact manifolds of dimension greater or equal to 6, for which the subgroups H_J^+, H_J^- may even have a non-trivial intersection. Dimension 4 is special, as it was proved in [11] that on any compact almost complex 4-manifold (M^4, J), the subgroups H_J^+, H_J^- do yield a direct sum decomposition for $H^2(M, \mathbb{R})$. In this paper, we still concentrate to dimension 4, and give some computations and estimates for the dimensions $h_J^{\pm}$ of the subgroups $H_J^{\pm}$.

After some preliminaries, section 2 contains a result of Lejmi [23] (Lemma 2.3 in our paper), from which one can see the space H_J^- as the kernel of an elliptic operator. Following an observation of Vestislav Apostolov, Lejmi's

lemma combined with a classical result of Kodaira and Morrow yields semi-continuity properties of $h^{\pm}_{J_t}$ for any path J_t of almost complex structures on a compact 4-manifold (Theorem 2.6). Two conjectures are made about the dimension h^-_J on a compact 4-manifold: namely, that h^-_J vanishes for generic almost complex structures (Conjecture 2.4), and that an almost complex structure with $h^-_J \geq 3$ is necessarily integrable (Conjecture 2.5).

In section 3, we confirm the first conjecture for 4-manifolds with $b^+ = 1$ (Theorem 3.1). The main topic of section 3 is the notion of metric related almost complex structures. Two almost complex structures are said to be *metric related* if they induce the same orientation and they admit a common compatible metric. We compute the subgroups H^+_J, H^-_J and their dimensions h^+_J, h^-_J for almost complex structures metric related to an integrable one. The main result is:

Theorem 1.1. *Let (M, J) be a compact complex surface. If $\tilde{J}$ is an almost complex structure on M metric related to J, $\tilde{J} \not\equiv \pm J$, then $h^-_{\tilde{J}} \in \{0, 1, 2\}$. The almost complex structures $\tilde{J}$ with $h^-_{\tilde{J}} = 0$ form an open and dense set with respect to the C^∞-topology in the space of almost complex structures metric related to J. The almost complex structures $\tilde{J}$ for which $h^-_{\tilde{J}} = 1$ or $h^-_{\tilde{J}} = 2$ are explicitly described. In particular, the case $h^-_{\tilde{J}} = 2$ appears only when (M, J) is a complex torus, or a K3 surface.*

A main tool in the proof of Theorem 1.1 are the Gauduchon metrics. We also use them to give an alternative proof of the fact first observed in [24], that for a compact complex surface (M, J), J is tamed by a symplectic form if and only if b_1 is even (Proposition 3.3). Section 3 ends with a couple of applications of Theorem 1.1. In Theorem 3.24, we prove that the intersection of H^+_J, H^-_J could be non-trivial even for a Kähler J if the compactness assumption is removed. Theorem 3.25 shows that the examples of non-integrable almost complex structures with $h^-_{\tilde{J}} = 2$ from Theorem 1.1 cannot admit a smooth pseudo-holomorphic blowup.

The so called *well-balanced* almost Hermitian 4-manifolds are introduced in section 4, as a natural generalization of both the Hermitian 4-manifolds and the almost Kähler ones. It is likely that this new notion has links with generalized complex geometry, but we leave the study of these possible links for future work. For now, we give some examples of well-balanced almost Hermitian 4-manifolds (Proposition 4.5), and prove a vanishing result for h^-_J on a well-balanced compact almost Hermitian 4-manifold with Hermitian Weyl tensor (Theorem 4.8).

Finally, in section 5 we discuss Donaldson's symplectic version of the Calabi-Yau equation on 4-manifolds. We observe that his technique based on the Implicit Function Theorem can also be used to obtain a stronger semi-continuity property for $h^{\pm}_J$ near a J which admits a compatible symplectic form (Theorem 5.4).

Acknowledgments: We are very grateful to V. Apostolov for pointing out Theorem 2.6 and for other valuable suggestions. We also thank D. Angella and A. Tomassini for good discussions and for sending us the preprint [2], and V. Tosatti for useful comments.

2. Definitions and preliminary results

Let (M, J) be an almost complex manifold. The almost complex structure J acts on the bundle of real 2-forms Λ^2 as an involution,
by $\alpha(\cdot,\cdot) \to \alpha(J\cdot, J\cdot)$, thus we have the splitting into J-invariant, respectively, J-anti-invariant 2-forms

$$\Lambda^2 = \Lambda^+_J \oplus \Lambda^-_J. \tag{1}$$

We will denote by Ω^2 the space of 2-forms on M (C^∞-sections of the bundle Λ^2) , Ω^+_J the space of J-invariant 2-forms, etc. For any $\alpha \in \Omega^2$, the J-invariant (resp. J-anti-invariant) component of α with respect to the decomposition (1) will be denoted by α' (resp. α''). We will also use the notation $\mathcal{Z}^2$ for the space of closed 2-forms on M and $\mathcal{Z}^\pm_J = \mathcal{Z}^2 \cap \Omega^\pm_J$ for the corresponding projections.

The bundle Λ^-_J inherits an almost complex structure, still denoted J, by

$$\alpha \in \Lambda^-_J \;\to\; J\alpha \in \Lambda^-_J, \text{ where } J\alpha(X,Y) = -\alpha(JX,Y).$$

It is well known that when J is integrable (in any dimension), we have

$$\beta \in \mathcal{Z}^-_J \Leftrightarrow J\beta \in \mathcal{Z}^-_J.$$

Conversely (see e.g. [27]), if (M, J) is a connected almost complex 4-manifold and there is a pair $\beta \in \mathcal{Z}^-_J, J\beta \in \mathcal{Z}^-_J$ (β not identically zero), then J is integrable.

The following definitions were introduced in [24] for an arbitrary almost complex manifold (M, J).

Definition 2.1. *(i) The J-invariant, respectively, J-anti-invariant cohomology subgroups H^+_J, H^-_J, are defined by*

$$H^\pm_J = \{\mathfrak{a} \in H^2(M;\mathbb{R}) | \exists\, \alpha \in \mathcal{Z}^\pm_J \text{ such that } [\alpha] = \mathfrak{a}\}\,;$$

(ii) J is said to be C^∞-pure if $H^+_J \cap H^-_J = \{0\}$;
(iii) J is said to be C^∞-full if $H^+_J + H^-_J = H^2(M;\mathbb{R})$;
(iv) J is C^∞-pure and full if $H^+_J \oplus H^-_J = H^2(M;\mathbb{R})$.

As noted in the introduction, when J is integrable and admits a compatible Kähler metric, or when (M, J) is a complex surface, the subgroups $H^\pm_J$ are nothing but the (real) Dolbeault cohomology groups (see [11], [1]):

$$H^+_J = H^{1,1}_{\bar\partial} \cap H^2(M;\mathbb{R}), \quad H^-_J = (H^{2,0}_{\bar\partial} \oplus H^{0,2}_{\bar\partial}) \cap H^2(M;\mathbb{R}). \tag{2}$$

In these cases, there is a weight 2 formal Hodge decomposition (more generally, this is true whenever the Fröhlicher spectral sequence degenerates at first step), so J is C^∞-pure and full. For complex dimensions greater

or equal to 3, there are known examples of complex structures for which the Fröhlicher spectral sequence does not degenerate at first step. Recently, Angella and Tomassini have also shown in [1] that Iwasawa manifold X^6 admits complex structures which are not C^∞-pure nor full. Other interesting examples appear in [2], showing, in particular, that the notions of C^∞-pure and C^∞-full are not related. The first 6-dimensional examples of (non-integrable) almost complex nilmanifolds which are not C^∞-pure nor full were given by Fino and Tomassini [12].

By contrast, in dimension 4, the following result was proved in [11]:

Theorem 2.2. *If M is a compact 4-dimensional manifold then any almost complex structure J on M is C^∞-pure and full, i.e.*

$$H^2(M;\mathbb{R}) = H^+_J \oplus H^-_J \ . \tag{3}$$

We refer to [11] for the proof of Theorem 2.2. It is based on Hodge theory and the particularity of dimension 4 stemming from the self-dual, anti-self-dual decomposition induced by the Hodge operator $*_g$ of a Riemannian metric g on M:

$$\Lambda^2 = \Lambda^+_g \oplus \Lambda^-_g . \tag{4}$$

If the metric g is compatible with the almost complex structure J and we let ω be the fundamental form defined by $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, the decompositions (1) and (4) are related by

$$\Lambda^+_J = \underline{\mathbb{R}}(\omega) \oplus \Lambda^-_g , \tag{5}$$

$$\Lambda^+_g = \underline{\mathbb{R}}(\omega) \oplus \Lambda^-_J . \tag{6}$$

In particular, any J-anti-invariant 2-form in 4-dimensions is self-dual, thus any closed, J-anti-invariant 2-form is harmonic, self-dual. This enables us to identify the space H^-_J with $\mathcal{Z}^-_J$, and further, with the set $\mathcal{H}^{+,\omega^\perp}_g$ of harmonic self-dual forms pointwise orthogonal to ω. In fact, it is an observation of Lejmi [23] that this space can be seen as the kernel of an elliptic operator defined on Ω^-_J.

Lemma 2.3. *([23], Lemma 4.1) Let (M^4, g, J, ω) be a compact, almost Hermitian 4-manifold. Consider the operator*

$$P : \Omega^-_J \to \Omega^-_J \ , \quad P(\psi) = (d\delta^g\psi)'' ,$$

where δ^g is the codifferential with respect to the metric g and the superscript $''$ denotes the projection $\Omega^2 \to \Omega^-_J$. Then P is a self-adjoint strongly elliptic linear operator with kernel the g-harmonic J-anti-invariant 2-forms.

Lemma 4.1 in [23] is stated for almost Kähler 4-manifolds, but the reader can easily check that its proof does not use the assumption that ω is closed.

Indeed, since $\Omega_J^- \subset \Omega_g^+$ and since the Riemannian Laplace operator $\Delta^g = d\delta^g + \delta^g d$ preserves the decomposition (4), note that for $\psi \in \Omega_J^-$,

$$P(\psi) = \frac{1}{2}\Delta^g\psi - \frac{1}{4} < \Delta^g\psi, \omega > \omega.$$

Then since ψ and ω are pointwise orthogonal, a short computation gives

$$< \Delta^g\psi, \omega > = -2\delta^g(< \psi, \nabla\omega >) + < \psi, \Delta^g\omega > .$$

The right side contains clearly only one derivative in ψ, and the lemma follows easily. Here and later in the paper, δ^g denotes the divergence operator, i.e. the adjoint of d with respect to the metric g.

Let us denote the dimension of $H_J^{\pm}$ by $h_J^{\pm}$, let b_2 be the second Betti number, and $b^{\pm}$ be the "self-dual", resp. "anti-self-dual" Betti numbers of the 4-manifold M. By Theorem 2.2 and the observations above, we have

$$h_J^+ + h_J^- = b_2; \tag{7}$$

$$h_J^+ \geq b^-, \quad h_J^- \leq b^+. \tag{8}$$

We propose the following two conjectures:

Conjecture 2.4. *For generic almost complex structures J on a compact 4-manifold M, $h_J^- = 0$.*

Conjecture 2.5. *On a compact 4-manifold, if $h_J^- \geq 3$ then J is integrable.*

In the case $b^+ = 1$, Conjecture 2.4 is proved in Theorem 3.1. Theorem 1.1 is a further partial answer and motivation for both conjectures.

We end this section by establishing a path-wise semi-continuity property for $h_J^{\pm}$ on a compact 4-manifold. This result was pointed out to the first author by Vestislav Apostolov.

Theorem 2.6. *Let M be a compact 4-manifold and let J_t, $t \in [0,1]$ be a smooth family of almost complex structures on M. Then $h_{J_t}^-$ (resp. $h_{J_t}^+$) is an upper-semi-continuous (resp. lower-semi-continuous) function in t. That is, for any $t \in [0,1]$ there exists $\epsilon > 0$ such that if $s \in [0,1]$, $|s-t| < \epsilon$,*

$$h_{J_s}^- \leq h_{J_t}^- \ , \ \ h_{J_s}^+ \geq h_{J_t}^+.$$

Proof. The statement about $h_{J_t}^-$ follows directly from Lemma 2.3 and a classical result of Kodaira and Morrow showing the upper-semi-continuity of the kernel of a family of elliptic differential operators (Theorem 4.3 in [19]). The statement on $h_{J_t}^+$ follows from Theorem 2.2. □

The following immediate corollary sheds some light on Conjecture 2.4 and on the density statement in Theorem 1.1.

Corollary 2.7. *If (M^4, J) is a compact almost complex manifold with $h_J^- = 0$ and J_t is a deformation of J, then for small t, $h_{J_t}^- = 0$.*

Remark 2.8. In Theorem 5.4, we establish a stronger semi-continuity property for $h_J^{\pm}$ near an almost complex structure which admits a compatible symplectic form. Theorems 2.6 and 5.4 are no longer true in higher dimension, as recent examples of Angella and Tomassini imply (see Propositions 4.1, 4.3 and Examples 4.2, 4.4 in [2]). Note that their Example 4.2 shows that the semi-continuity property fails in dimensions higher than 4, even if one has a path of almost complex structures which are C^∞-pure and full.

3. Computations of h_J^-

3.1. **Generic vanishing of h_J^- when $b^+ = 1$.** In this subsection we confirm Conjecture 2.4 when $b^+ = 1$.

Theorem 3.1. *Suppose M is a compact $4-$manifold with $b^+ = 1$ admitting almost complex structures. The almost complex structures J with $h_J^- = 0$ form an open and dense subset in the set of all almost complex structures on M, with C^∞-topology.*

3.1.1. *Topology of the space of almost complex structures.* Let $\mathcal{J} = \mathcal{J}^\infty$ be the space of C^∞ almost complex structures. Let us first describe the C^∞ topology of $\mathcal{J}^\infty$.

It is well known that the space $\mathcal{J}^l$ of C^l almost complex structures has a natural separable Banach manifold structure via the C^l norm (see [25] for example). The natural C^∞ topology on $\mathcal{J}^\infty$ is induced by the sequence of semi-norms $C^0, C^1, \cdots, C^l, \cdots$. Locally, near a C^∞ almost complex structure J, $\mathcal{J}$ is a subspace of $\mathcal{J}^l$ with finer topology.

With the C^∞ topology, $\mathcal{J} = \mathcal{J}^\infty$ is a Fréchet manifold. A complete metric inducing the C^∞ topology on it can be defined by

$$d(x, y) = \sum_{k=0}^{\infty} \frac{\|x - y\|_k}{1 + \|x - y\|_k}\, 2^{-k}. \tag{9}$$

Here $\|\cdot\|_k$ represents the C^k semi-norm on it.

3.1.2. *The space of $g-$compatible almost complex structures.* To prove the density statement in Theorem 3.1 we need to consider the space of almost complex structures compatible with a fixed Riemannian metric g. This can be described as the space of g-self-dual 2-forms ω satisfying $|\omega|_g^2 = 2$ pointwise on M (equivalently, the space of smooth sections of the twistor bundle associated to (M, g)). The C^∞-topology on this subspace corresponds to C^∞-topology on the space of 2-forms.

Suppose we also fix a $g-$compatible pair (J, ω). Then any $g-$compatible almost complex structure corresponds to a $2-$form

$$\tilde{\omega} = f\omega + \beta, \text{ with } \beta \in \Omega_J^-,\ f \in C^\infty(M) \text{ so that } 2f^2 + |\beta|^2 = 2. \tag{10}$$

For us, the following variation will be useful, extending an idea from [21]. Suppose further a section $\alpha \in \Omega_J^-$ is given. One can define new g-compatible

almost complex structures as follows: pick smooth functions f and r on M so that the form

$$\tilde{\omega} = f\omega + r\alpha \tag{11}$$

satisfies $|\tilde{\omega}|_g^2 = 2$, and let $\tilde{J}$ be the almost complex structure defined by $(g, \tilde{\omega})$. Equivalently, f and r should satisfy the pointwise condition

$$2f^2 + r^2|\alpha|_g^2 = 2. \tag{12}$$

For any $\alpha \in \Omega_J^-$, one can find such functions f and r. For instance, take r to be small enough so that $r^2|\alpha|_g^2 < 2$ everywhere; then f is determined up to sign by $f = \pm(1 - \frac{1}{2}r^2|\alpha|_g^2)^{1/2}$. Junho Lee's almost complex structures J_α (see [21]) are obtained for the specific choice [1]

$$r = \frac{4}{2 + |\alpha|^2} \ \text{ and } \ f = \frac{2 - |\alpha|^2}{2 + |\alpha|^2}\ . \tag{13}$$

Note that we actually get a pair of almost complex structures $J_\alpha^\pm$, as for the above choice of r, we have the sign freedom in choosing f. Junho Lee defines these almost complex structures on a Kähler surface (M, J, g) and uses them as a tool for an easier computation of the Gromov-Witten invariants. Particularly important in his work are the almost complex structures J_α corresponding to *closed* α's, i.e $\alpha \in \mathcal{Z}_J^-$.

Another natural choice for (r, f) is

$$r = \pm f = \pm\frac{\sqrt{2}}{\sqrt{2 + |\alpha|^2}}\ . \tag{14}$$

This corresponds to almost complex structures that arise from the forms $\pm\omega + \alpha$, conformally rescaled to satisfy the norm condition.

Even more generally, given α, we may choose r so that $r^2|\alpha|_g^2 \leq 2$, with equality at some points, but then at such points we have to require the smoothness of the function $(1 - \frac{1}{2}r^2|\alpha|_g^2)^{1/2}$. Note also that if such points exists, then we no longer have an "up to sign choice" for f overall.

Finally, note that we can (and will) choose r to satisfy $r^2|\alpha|_g^2 < 2$ and be supported on a small open set in M. Then, for $f = (1 - \frac{1}{2}r^2|\alpha|_g^2)^{1/2}$, the new almost complex structure $\tilde{J}$ coincides with J outside the support of r.

3.1.3. *Proof of Theorem 3.1.*

Proof. First we show the density. Let J be an almost complex structure on M. It follows from (6) that $h_J^- \in \{0, 1\}$. If $h_J^- = 0$, Corollary 2.7 shows that in any neighborhood of J there are other almost complex structures $\tilde{J}$

[1] There is a factor "2" difference in the convention for the norm of a two form between our paper and [21]. For us, if (g, J, ω) is a 4-dimensional almost Hermitian structure, $|\omega|_g^2 = 2$, whereas in [21], $|\omega|_g^2 = 1$. This explains the apparent difference between our r and f and those in Proposition 1.5 in [21].

with $h^-_{\tilde{J}} = 0$. If $h^-_J = 1$, let $\alpha \in \mathcal{Z}^-_J$, normalized so that $\int_M \alpha^2 = 1$. Pick a J-compatible metric g and let ω be the fundamental form associated to (g, J). The form α is g-harmonic and point-wise orthogonal to ω. As in (11), let $\tilde{\omega} = f\omega + r\alpha$, for some functions f, r satisfying (12), and define $\tilde{J}$, the almost complex structure induced by $(g, \tilde{\omega})$. If $r \not\equiv 0$, it is clear that $h^-_{\tilde{J}} = 0$, as $\tilde{\omega}$ is no longer point-wise orthogonal to α. On the other hand, we can choose r to be compactly supported on a small set, so $\tilde{J}$ can be arbitrarily close to J.

For openness, we prove that the complement is closed. Let J_k be a sequence of almost complex structures with $h^-_{J_k} = 1$ converging to the almost complex structure J. Let g be a J-compatible Riemannian metric and let

$$g_k(\cdot, \cdot) = \frac{1}{2}(g(\cdot, \cdot) + g(J_k \cdot, J_k \cdot)).$$

Clearly, g_k is a Riemannian metric compatible with J_k and (g_k, J_k) converges to (g, J). Denote by Δ^k the Hodge-DeRham Laplace operator associated to g_k and by $\mathbb{G}^k$ the Green operator associated to Δ^k.

Let ψ be a non-zero g-harmonic, self-dual two form, normalized so that $\int_M \psi^2 = 1$ (up to sign, ψ is unique with these properties, as $b^+ = 1$).

Consider the Hodge decomposition of the 2-form ψ with respect to each of the metrics g_k.

$$\psi = (\psi - \mathbb{G}^k(\Delta^k \psi)) + \mathbb{G}^k(\Delta^k \psi) = \psi_{h,k} + \psi_{ex,k},$$

where $\psi_{h,k} = \psi - \mathbb{G}^k(\Delta^k \psi)$ denotes the g_k-harmonic part of ψ and $\psi_{ex,k} = \mathbb{G}^k(\Delta^k \psi)$ is the g_k-exact part of ψ. Since $g_k \to g$ and $\Delta^g \psi = 0$, this implies

$$\psi_{h,k} \to \psi \quad , \quad \psi_{ex,k} \to 0, \text{ as } k \to \infty.$$

Moreover, if $(\psi_{k,h})^+$ denotes the g_k-self-dual part of $\psi_{k,h}$, we have

$$(\psi_{k,h})^+ \to \psi \ .$$

But since $b^+ = h^-_{J_k} = 1$, the g_k-harmonic, self-dual forms $(\psi_{k,h})^+$ are J_k-anti-invariant. Since $J_k \to J$, it follows that ψ must be J-anti-invariant. Thus, $h^-_J = 1$. □

Remark 3.2. There exist compact almost complex 4-manifolds (M, J) with $b^+ = 1$ and $h^-_J = 1$. Proposition 6.1 of [12] contains one such example (see also Proposition 4.5 (iii) in this paper, where this example appears in a different context). Note also that any such almost complex structure cannot be tamed by a symplectic form, as a consequence of Theorem 3.3 of [11].

3.2. **When J is integrable.** If (M^4, J) is a compact complex surface, it follows from (2) that $h^\pm_J$ are the same as the dimensions of the corresponding Dolbeault groups

$$h^+_J = h^{1,1}_{\bar{\partial}}, \quad h^-_J = 2h^{2,0}_{\bar{\partial}}. \tag{15}$$

Together with the signature theorem (Theorem 2.7 in [9]), we get

$$h_J^+ = \begin{cases} b^- + 1 & \text{if } b_1 \text{ even} \\ b^- & \text{if } b_1 \text{ odd}, \end{cases} \qquad h_J^- = \begin{cases} b^+ - 1 & \text{if } b_1 \text{ even} \\ b^+ & \text{if } b_1 \text{ odd}. \end{cases} \tag{16}$$

It is a deep, but now well known fact that the cases b_1 even/odd correspond to whether the complex surface (M, J) admits or not a compatible Kähler structure. We observe that there is a more direct proof for the following weaker statement.

Proposition 3.3. *Let (M, J) be a compact complex surface. The following are equivalent:*

(i) b_1 is even; (ii) $b^+ = h_J^- + 1 = 2h_{\bar\partial}^{2,0} + 1$; (iii) J is tamed.

Similarly, the following are equivalent:

(i') b_1 is odd; (ii') $b^+ = h_J^- = 2h_{\bar\partial}^{2,0}$; (iii') J is not tamed.

An almost complex structure J is said to be *tamed* if there exists a symplectic form ω such that $\omega(X, JX) > 0$ for any non-zero tangent vector X. The tame-compatible question of Donaldson [10] predicts that on a compact 4-manifold any tame almost complex structure J admits, in fact, a *compatible* symplectic form, that is, a symplectic form $\tilde\omega$, so that $\tilde\omega(\cdot, J\cdot)$ is a Riemannian metric.

It was first observed in [24] using a result of [17], that on a compact complex surface the tame condition is equivalent with b_1 even. Proposition 3.3 gives a different proof of this fact. Assuming Kodaira's classification, the tame condition is thus equivalent with the compatibility. As Donaldson points out, a direct confirmation of the tame-compatible question would lead to a different proof of the fact that b_1 even corresponds to a complex surface of Kähler type. At least in the case $b^+ = 1$, the tame-compatible question is known to be a consequence of the symplectic Calabi-Yau problem, also introduced by Donaldson in [10] (see also [31], [29], [28], and section 5 below).

A key tool in our proof of Proposition 3.3 are the *Gauduchon metrics* whose definition we recall next.

3.2.1. *Gauduchon metric.* For an (almost) Hermitian manifold (M, g, J, ω), the *Lee form* θ is defined by $\theta = J\delta^g\omega$, or, equivalently in dimension 4, by $d\omega = \theta \wedge \omega$. It is well known that $d\theta$ is a conformal invariant. When J is integrable, the case when θ is closed (exact) corresponds to locally (globally) conformal Kähler metrics. Obviously, Hermitian metrics with $\theta = 0$ are, in fact, Kähler metrics.

Definition 3.4. *A Hermitian metric such that the Lie form is co-closed, i.e. $\delta^g\theta = 0$, is called a Gauduchon metric (or standard Hermitian metric, in the original terminology of* [13]*).*

The existence and uniqueness (up to homothety) of a Gauduchon metric in each conformal class is shown in [13]. The result is much more general; it

does not require integrability, nor restriction to dimension 4. For us, the key property of a (Hermitian) Gauduchon metric in dimension 4 is the following:

Proposition 3.5. *([14]) On a compact complex surface M endowed with a Gauduchon metric g, the trace of a harmonic, self-dual form is a constant.*

For the proof of Proposition 3.5, we refer the reader to Lemma II.3 in [14] (see also [6], Proposition 3, for a slightly different argument). The Proposition 3.5 implies that for Hodge decomposition arguments, the Gauduchon metrics behave quite like the Kähler ones. This simple fact yields good consequences.

3.2.2. *Proof of Proposition 3.3.*

Proof. As we mentioned already (and is easy to check), for a complex surface the groups $H_J^{\pm}$ are identified with the (real) Dolbeault groups as in (2). Using (8), we thus have

$$b^+ \geq h_J^- = 2h_{\bar\partial}^{2,0}.$$

We'll show (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

It is well known that for any almost complex 4-manifold, $b_1 + b^+$ is odd, thus (ii) $\Rightarrow$ (i) is obvious.

Now assume (i), which is equivalent with b^+ odd, by the above observation. It follows that $b^+ > h_J^- = 2h_{\bar\partial}^{2,0}$. Choose a J-compatible conformal class and let g be the Gauduchon metric with total volume one in this class; denote by ω the fundamental 2-form induced by (g, J). Let ψ be a non-trivial harmonic self-dual 2-form, whose cohomology class $[\psi]$ is cup-product orthogonal to H_J^- (such ψ exists because $b^+ > h_J^-$). From Proposition 3.5 and (6), ψ decomposes as

$$\psi = a\,\omega + \beta, \text{ with } a \text{ constant and } \beta \in \Omega_J^- \ . \tag{17}$$

The constant a is non-zero, by the assumption that $[\psi]$ is cup product orthogonal to H_J^-. This implies right away that ψ is symplectic (as β is self-dual and point-wise orthogonal to ω). By eventually replacing ψ by $-\psi$, we can assume also that $a > 0$, so J is tamed (by ψ or $-\psi$). Thus, we proved (i) $\Rightarrow$ (iii).

Next, suppose that ψ is a symplectic form that tames J. As pointed out in [10], $\mathbb{R}\,\psi + \Lambda_J^-$ is is a 3-dimensional bundle on M, positive-definite with respect to the wedge pairing and the volume form ψ^2. This induces a J-compatible conformal class. Let g be the Gauduchon metric in this class and denote again by ω the fundamental form of (g, J). The form ψ is g-self-dual and closed, thus it is harmonic. Then relation (17) holds, with $a > 0$, by the assumption that ψ tames J. It follows that $[\psi] \notin H_J^-$, thus $b^+ > h_J^-$. Now assume that ψ_1 and ψ_2 are harmonic self-dual 2-forms, whose cohomology classes $[\psi_1]$, $[\psi_2]$ are cup-product orthogonal to H_J^-. As above,

$$\psi_1 = a_1\,\omega + \beta_1 \ , \quad \psi_2 = a_2\,\omega + \beta_2,$$

with a_1, a_2 non-zero constants and $\beta_1, \beta_2 \in \Omega^-_J$. But then $a_2\,\psi_1 - a_1\,\psi_2 = a_2\,\beta_1 - a_1\,\beta_2$ is J-anti-invariant and closed. Together with the assumptions that $[\psi_1]$, $[\psi_2]$ are cup-product orthogonal to H^-_J, this can happen only if

$$a_2\,\psi_1 - a_1\,\psi_2 \equiv 0\ .$$

Thus $b^+ - h^-_J = 1$, so (iii) $\Rightarrow$ (ii) is proved.

Remark that the proof shows that (i), (ii), (iii) are also equivalent to (iv) $b^+ > h^-_J$. The equivalence of (i'), (ii'), (iii') is then the negation of the above. □

3.3. **Comparing metric related almost complex structures.**

Notice that when J is integrable the dimensions $h^\pm_J$ are topological invariants. Such a property is certainly no longer true for general almost complex structures. However, we are still able to calculate the exact value of $h^\pm_J$ for almost complex structures which are metric related to integrable ones. To achieve this we first derive some general results about metric related almost complex structures.

3.3.1. *Estimates for $g-$related almost complex structures.* We again fix a Riemannian metric g.

Definition 3.6. *Suppose J and $\tilde{J}$ are two almost complex structures inducing the same orientation on a 4-manifold M. J and $\tilde{J}$ are said to be $g-$related if they are both compatible with g.*

It is clear that if g has this property, then so does any metric from its conformal class. Also, if J and $\tilde{J}$ are $g-$related then

$$\Lambda^-_J + \Lambda^-_{\tilde{J}} \subset \Lambda^+_g, \text{ and hence } H^-_J + H^-_{\tilde{J}} \subset \mathcal{H}^+_g.$$

Recall that since any closed J-anti-invariant form is harmonic, self-dual, we can identify H^-_J with $\mathcal{Z}^-_J$ and see it as a subspace of $\mathcal{H}^+_g$ (the space of harmonic, self-dual forms).
The following observation is the key for the computations of $h^\pm_J$ we achieve in this section.

Proposition 3.7. *Suppose J and $\tilde{J}$ are $g-$related almost complex structures on a connected 4-manifold M, with $\tilde{J} \not\equiv \pm J$. Then* $\dim\,(H^-_J \cap H^-_{\tilde{J}}) \leq 1$.

Proof. Let ω and $\tilde{\omega}$ be the corresponding self-dual 2-forms. By assumption, the set

$$U = \{p \in M\ |J(p) \neq \pm\tilde{J}(p)\} = \{p\ |\dim\,(\mathrm{Span}\{\omega(p), \tilde{\omega}(p)\}) = 2\}$$

is a non-empty open set in M. Without loss of generality we can assume that U is connected. Otherwise, we can make the reasoning below on a connected component of U.

Assume $H^-_J \cap H^-_{\tilde{J}} \neq \{0\}$ and let $\alpha_1, \alpha_2 \in \mathcal{Z}^-_J \cap \mathcal{Z}^-_{\tilde{J}} = H^-_J \cap H^-_{\tilde{J}}$, not identically zero. Let U' be the open subset of U where neither α_1 or α_2 vanishes. $U' \neq \emptyset$ because α_1 and α_2 are g(-self-dual)-harmonic forms, thus

they satisfy the unique continuation property. Since on U', Span$\{\omega, \tilde{\omega}\}$ is a 2-dimensional subspace of $\Lambda^+_g M$ and α_1, α_2 are both orthogonal to this subspace, there exists $f \in C^\infty(U')$ such that $\alpha_2 = f\alpha_1$. Since α_1, α_2 are, by assumption, both closed, it follows that $0 = df \wedge \alpha_1$. But α_1 is non-degenerate on U' (it is self-dual, non-vanishing). Thus $df = 0$, so $f = const.$ on U'. It follows that $\alpha_2 = const.\ \alpha_1$ on U', but, by unique continuation, this holds on the whole M. □

Remark 3.8. The estimate in Proposition 3.7 is sharp. Indeed, let (M, g, J, ω) be a connected almost Hermitian 4-manifold, and assume that $\alpha \in \mathcal{Z}^-_J$ is not identically zero. Consider a g-compatible almost complex structure $\tilde{J}$, arising from a self-dual 2-form

$$\tilde{\omega} = f\omega + rJ\alpha \ , \tag{18}$$

where f and r are C^∞-functions, so that

$$|\tilde{\omega}|^2_g = 2f^2 + r^2|\alpha|^2_g = 2 \ . \tag{19}$$

By (6) applied to $(g, \tilde{J}, \tilde{\omega})$, observe that α is $\tilde{J}-$anti-invariant. Hence, by Proposition 3.7, $H^-_J \cap H^-_{\tilde{J}} = \mathrm{Span}([\alpha])$. Conversely, any g-compatible $\tilde{J}$ such that $[\alpha] \in H^-_J \cap H^-_{\tilde{J}}$ will have a fundamental form $\tilde{\omega}$ given by (18) at least on the open dense set $M' = M \setminus \alpha^{-1}(0)$, with functions $f, r \in C^\infty(M')$ satisfying (19).

Observe that compactness is not needed for Proposition 3.7 or Remark 3.8. In the compact case, Proposition 3.7 has the following easy consequence.

Corollary 3.9. *In the space of almost complex structures compatible to a given metric g on a compact 4-manifold, there is at most one J such that*

$$h^-_J \geq \begin{cases} \frac{b^++3}{2} & \text{if } b^+ \text{ is odd} \\ \frac{b^++2}{2} & \text{if } b^+ \text{ is even.} \end{cases} \tag{20}$$

3.3.2. *Metric related almost complex structures.*

Definition 3.10. *Two almost complex structures J and $\tilde{J}$ on a 4-manifold M are said to be metric related if they induce the same orientation and are $g-$related for some Riemannian metric g on M.*

If we fix a volume form σ on M, two almost complex structures J and $\tilde{J}$ are metric related if and only if there exists a 3-dimensional sub-bundle $\Lambda^+ \subset \Lambda^2 M$, positive definite with respect to the wedge pairing and σ, such that $\Lambda^-_J \subset \Lambda^+$, $\Lambda^-_{\tilde{J}} \subset \Lambda^+$. One important difference versus the "g-related" condition for a fixed g is that the metric related condition is not transitive. Because of this, Corollary 3.9, for instance, is not automatically clear under just the metric related assumptions. However, Proposition 3.7 clearly extends to the metric related case. One immediate consequence is:

Corollary 3.11. *Suppose J and $\tilde{J}$ are metric-related almost complex structures on a compact 4-manifold M, with $\tilde{J} \not\equiv \pm J$.*

(i) If $h_J^- = b^+$, then $h_{\tilde{J}}^- \leq 1$.

(ii) If $h_J^- = b^+ - 1$, then $h_{\tilde{J}}^- \leq 2$.

From this Corollary, one obtains immediately the claim $h_{\tilde{J}}^- \in \{0, 1, 2\}$ from the statement of Theorem 1.1. The results in the next subsections will be more specific about when each case occurs.

3.4. **Proof of Theorem 1.1.** Throughout this subsection, unless stated otherwise, J will denote a *complex* structure on a compact 4-manifold M. Denote by $\mathcal{J}$ the space of all (smooth) almost complex structures on M and by $\mathcal{J}_J$ the set of almost complex structures which are metric related to the fixed J. On both spaces $\mathcal{J}$ and $\mathcal{J}_J$ we consider the C^∞-topology. For reasons that will be apparent soon, it is best to divide the proof into some cases depending on the type of the surface (M, J).

3.4.1. *Surfaces of non-Kähler type, or of Kähler type but with non-trivial canonical bundle.* For these we have the following result.

Theorem 3.12. *Let (M, J) be a compact complex surface of non-Kähler type, or a compact complex surface of Kähler type, but with topologically non-trivial canonical bundle. If $\tilde{J} \in \mathcal{J}_J$, $\tilde{J} \not\equiv \pm J$, then either (i) $h_{\tilde{J}}^- = 0$, or (ii) $h_{\tilde{J}}^- = 1$. Case (i) occurs for an open, dense set of almost complex structures in $\mathcal{J}_J$. Case (ii) occurs precisely when there exist $\alpha \in \mathcal{Z}_J^-$ such that $H_{\tilde{J}}^- = \mathrm{Span}([\alpha])$, so these $\tilde{J}$ appear as described in Remark 3.8.*

Proof. First, we justify the statement $h_{\tilde{J}}^- \in \{0, 1\}$. For a complex surface of non-Kähler type, this follows directly from Corollary 3.11. Now suppose that (M, J) is a complex surface of Kähler type with topologically non-trivial canonical bundle. Consider the conformal class of metrics compatible with both J and $\tilde{J}$ and let g be the Gauduchon metric with respect to J in this class. Let ω and $\tilde{\omega}$ denote the fundamental forms of (g, J) and $(g, \tilde{J})$, respectively. They are related as in (10),

$$\tilde{\omega} = f\omega + \beta, \text{ with } \beta \in \Omega_J^-, \ f \in C^\infty(M) \text{ so that } 2f^2 + |\beta|^2 = 2.$$

Suppose $h_{\tilde{J}}^- \neq 0$ and let $\psi \in \mathcal{Z}_{\tilde{J}}^-$, not identically zero. Since ψ is g-harmonic, from Proposition 3.5 it must be of the form $\psi = a\omega + \alpha$, with a constant and $\alpha \in \Omega_J^-$. The pointwise condition $< \psi, \tilde{\omega} >= 0$ is equivalent to

$$2af + < \alpha, \beta >= 0 \text{ everywhere on } M.$$

But β (and α) must vanish somewhere on M, since the canonical bundle is topologically non-trivial. At a point p where $\beta(p) = 0$, we have $f^2(p) = 1 \neq 0$, thus it follows that $a = 0$. Thus $\psi = \alpha$, but since $d\psi = 0$, it follows that $\psi = \alpha \in \mathcal{Z}_J^-$. Hence, $H_{\tilde{J}}^- \subset H_J^-$. The statement $h_{\tilde{J}}^- \in \{0, 1\}$ follows now

from Proposition 3.7. Note that we also proved the description of the case $h^-_{\tilde J} = 1$.

Next, we prove the density statement in Theorem 3.12. This follows from Corollary 2.7 and the following observation.

Proposition 3.13. *Let (M, J) be a compact complex surface as in Theorem 3.12. If $\tilde J \in \mathcal{J}_J$ and $h^-_{\tilde J} \neq 0$, then there exists $\tilde J' \in \mathcal{J}_J$, arbitrarily close to $\tilde J$, and with $h^-_{\tilde J'} = 0$.*

Proof. Suppose first that the geometric genus vanishes. For non-Kähler type, this means $b^+ = 2h^{2,0}_{\bar\partial} = 0$, so it follows from (6) that $h^-_{\tilde J} = 0$ for any $\tilde J$ on M (even not metric related to J). If (M, J) is of Kähler type and has zero geometric genus, then $h^-_J = 2h^{2,0}_{\bar\partial} = 0$, so the first part of the proof of Theorem 3.12 shows that $h^-_{\tilde J} = 0$, for any $\tilde J \in \mathcal{J}_J$.

Suppose next that the geometric genus of (M, J) does not vanish. Consider first the case $\tilde J \not\equiv \pm J$. From the first part of the proof of Theorem 3.12, the assumption $h^-_{\tilde J} \neq 0$ implies that there exists $\alpha \in \mathcal{Z}^-_J$ such that $H^-_{\tilde J} = \mathrm{Span}\{[\alpha]\}$. Moreover, there is a metric g on M compatible with both J and $\tilde J$ so that the corresponding forms ω and $\tilde\omega$ are related on $M' = M \setminus \alpha^{-1}(0)$ as in (18):

$$\tilde\omega = f\omega + rJ\alpha \ ,$$

where f and r are C^∞-functions on M', satisfying the norm condition (19). Note that even if the above relation is valid on the (open, dense) set M', $\tilde\omega$ is defined on the whole M. We deform $\tilde\omega$ as follows. Let $\tilde r$ be a compactly supported function on a small open subset U of M' and define

$$\tilde\omega' = \tilde f\tilde\omega + \tilde r\alpha \ ,$$

where the function $\tilde f$ is chosen so that $|\tilde\omega'|^2 = 2$. Let $\tilde J'$ be the almost complex structure induced by $(g, \tilde\omega')$. We claim that $h^-_{\tilde J'} = 0$.

Indeed, if $h^-_{\tilde J'} \neq 0$, as in the proof of Theorem 3.12, there exists $\beta \in \mathcal{Z}^-_J(M)$, so that $H^-_{\tilde J'} = \mathrm{Span}\{[\beta]\}$. Moreover, there are functions h, q so that

$$\tilde\omega' = h\omega + qJ\beta,$$

on the open dense set $M'' = M \setminus \beta^{-1}(0)$. On the other hand, on M' we have

$$\tilde\omega' = (\tilde f f)\omega + (\tilde f r)J\alpha + \tilde r\alpha \ .$$

It follows that on $M' \cap M''$, we have

$$qJ\beta = (\tilde f r)J\alpha + \tilde r\alpha \ .$$

Since $\tilde r$ is compactly supported on a small subset in M', it follows that $J\alpha$ and $J\beta$ are conformal multiples of one another on a non-empty open set. By the argument in the proof of Proposition 3.7, it follows that α and β are (non-zero) scalar multiples of one another on the whole M. Thus,

$H^-_{\tilde{J}'} = \mathrm{Span}\{[\alpha]\}$. But, by construction, on the set where $\tilde{r} \neq 0$, the form $\tilde{\omega}'$ is not point-wise orthogonal to α. Thus, $h^-_{\tilde{J}'} = 0$, as claimed.

In the case $\tilde{J} \equiv \pm J$, the argument is similar. We have even larger freedom in considering the deformation. Let $\alpha \in \mathcal{Z}^-_J$, and let r_1, r_2 be compactly supported on disjoint open sets. Consider

$$\tilde{\omega}' = f\omega + r_1\alpha + r_2 J\alpha,$$

where f is chosen to fulfill the norm condition. As above, one can show that $h^-_{\tilde{J}'} = 0$. □

Remark 3.14. The first part of the argument above shows the following: suppose (M, J) is a compact complex surface of Kähler type with vanishing geometric genus and topologically non-trivial canonical bundle. Then for any $\tilde{J} \in \mathcal{J}_J$, $h^-_{\tilde{J}} = 0$.

Finally, the openness statement in Theorem 3.12 follows from:

Proposition 3.15. *With the notations and assumptions of Theorem 3.12, suppose $\tilde{J}_k$ is a sequence of almost complex structures converging to $\tilde{J}$ (in the C^∞-topology), with $\tilde{J}_k, \tilde{J} \in \mathcal{J}_J$. If $h^-_{\tilde{J}_k} \neq 0$, then $h^-_{\tilde{J}} \neq 0$.*

Proof. The assumption $h^-_{\tilde{J}_k} \neq 0$ and the earlier arguments in the proof of Theorem 3.12, show that there exists $\alpha_k \in \mathcal{Z}^-_J$, such that $H^-_{\tilde{J}_k} = \mathrm{Span}([\alpha_k])$. We can normalize α_k so that $[\alpha_k] \cdot [\alpha_k] = 1$, where $\cdot$ denotes here the cup-product of cohomology. Thus, as α_k is a sequence on the unit sphere in $\mathcal{Z}^-_J$ which is a compact set (note that $\mathcal{Z}^-_J$ is finite dimensional), we can extract a subsequence, still denoted α_k, which converges to $\alpha \in \mathcal{Z}^-_J$. Obviously, $[\alpha] \neq 0$, as $[\alpha] \cdot [\alpha] = 1$. Moreover, since $\tilde{J}_k \to \tilde{J}$, $\alpha_k \to \alpha$, the relation

$$\alpha_k(\tilde{J}_k X, \tilde{J}_k Y) = -\alpha_k(X, Y)$$

implies

$$\alpha(\tilde{J}X, \tilde{J}Y) = -\alpha(X, Y).$$

Thus, $h^-_{\tilde{J}} \neq 0$. □

This also completes the proof of Theorem 3.12. □

Remark 3.16. A similar argument to the one in Proposition 3.15 yields the following result: given a metric g on a compact 4-manifold M, the set of g-compatible almost complex structures $\tilde{J}$ with $h^-_{\tilde{J}} = 0$ is open in the set of all g-compatible almost complex structures.

Remark 3.17. Under the assumptions of Theorem 3.12, if $\alpha \in \mathcal{Z}^-_J$, then the almost complex structures $\tilde{J}$ defined by (11) have $h^-_{\tilde{J}} = 1$, for any choice of (r, f) satisfying (12). In particular this is true for Junho Lee's almost complex structures $J^\pm_\alpha$ defined by (13). Note that since J is integrable,

$\alpha + iJ\alpha$ is a holomorphic $(2,0)$ form on M, hence the zero set $\alpha^{-1}(0)$ is a canonical divisor on (M,J).

Remark 3.18. If a compact 4-manifold M admits a pair of integrable complex structure (J_1, J_2) which are metric related then M has a bi-Hermitian structure. The study of such structures has been active recently, (see, for instance, [18] and the references therein), especially due to the link with generalized Kähler geometry ([16]). An easy consequence of Theorem 3.12 is the observation that a compact 4-manifold M with $b^+ = 2$, or $b^+ \geq 4$ does not admit a bi-Hermitian structure (compatible with the given orientation). This is not new, as it is easily seen from the classification results of [7] and [3], that manifolds admitting bi-Hermitian structures must have $b^+ \in \{0,1,3\}$.

3.4.2. *Surfaces of Kähler type with topologically trivial, but holomorphically non-trivial canonical bundle.*

Proposition 3.19. *Suppose that (M,J) is a complex surface of Kähler type with topologically trivial, but holomorphically non-trivial canonical bundle. Then for any almost complex structure $\tilde{J}$ on M (not even metric related to J), we have $h^-_{\tilde{J}} \in \{0,1\}$. The set of almost complex structures with $h^-_{\tilde{J}} = 0$ is open and dense with respect to the C^∞-topology, both in $\mathcal{J}$ and $\mathcal{J}_J$.*

Proof. Any such surface is a hyperelliptic surface. In this case $b^+ = 1$ and the claims follow from (6) and Theorem 3.1. ☐

We wonder whether the result in Remark 3.14 still holds in this case; in other words, is it still true that $h^-_{\tilde{J}} = 0$ for any $\tilde{J} \in \mathcal{J}_J$?

3.4.3. *Surfaces with holomorphically trivial canonical bundle.* Even if the non-Kähler subcase is covered by Theorem 3.12, it is worth considering it separately, as the result takes a very simple form. Surfaces of non-Kähler type with holomorphically trivial canonical bundles are Kodaira surfaces.

Thus, let (M,J) be a Kodaira surface. We have $h^-_J = b^+ = 2$. Let $\Phi = \beta + iJ\beta$ be a a nowhere vanishing holomorphic $(2,0)-$form trivializing the canonical bundle. The real and imaginary parts of Φ, β and $J\beta$ are both closed, nowhere vanishing J-anti-invariant forms. Suppose that g is a metric compatible with J and let ω be the corresponding non-degenerate form of (g,J). The triple $\{\omega, \beta, J\beta\}$ is a pointwise orthogonal basis of the rank 3 bundle Λ^+_g. Thus, any almost complex structure compatible with g corresponds to a form

$$\omega_{f,l,s} = f\omega + l\beta + sJ\beta, \tag{21}$$

where the functions $f,l,s \in C^\infty(M)$ satisfy $2f^2 + |\beta|^2(l^2+s^2) = 2$. We denote the almost complex structure corresponding to $(g, \omega_{f,l,s})$ by $J_{f,l,s}$. Every almost complex structure metric related to J can be obtained this way. Since for a Kodaira surface $\mathcal{H}^+_g = H^-_J = \mathrm{Span}(\alpha, J\alpha)$, the only possible

self-dual harmonic forms are of type

$$a\,\beta + b\,J\beta, \qquad \text{where } a \text{ and } b \text{ are constants.}$$

The only condition for this form lying in $H^-_{J_{f,l,s}}$ is $al + bs = 0$. Thus we have proved

Proposition 3.20. *If (M, g, J) is a Kodaira surface with a compatible metric g, using the notations above,*

$$h^-_{J_{f,l,s}} = 2 - rank(\mathrm{Span}(l, s)).$$

Clearly, $h^-_{J_{f,l,s}} = 0$ is the generic case, $h^-_{J_{f,l,s}} = 2$ if and only if $l = s = 0$, i.e. $\tilde{J} = \pm J$, and $h^-_{J_{f,l,s}} = 1$ if and only if the functions l and s are scalar multiples of each other, not both identically zero.

Next, suppose that (M, J) is a Kähler surface with holomorphically trivial canonical bundle. Then $b^+ = 3$ and (M, J) is a K3 surface or $4-$torus. As in the Kodaira surface case, let $\Phi = \beta + iJ\beta$ be a a nowhere vanishing holomorphic $(2, 0)-$form trivializing the canonical bundle. Consider a conformal class of metrics compatible with J, and, in this class, let g be the Gauduchon metric, with ω being the associated form. As above, denote by $J_{f,l,s}$ the almost complex structure corresponding to form $\omega_{f,l,s}$ given by (21). Every almost complex structure metric related to J is of the type $J_{f,l,s}$ for some Gauduchon metric g and for some functions f, l, s.

The difference from the Kodaira surface case is that $b^+ = \dim(\mathcal{H}^+_g) = 3$, rather than 2. As argued in Theorem 3.12, any $g-$harmonic form has a constant inner product with ω. Let ω' be the unique g-self-dual harmonic form with $< \omega', \omega >_g = 2$ and which is cup-product orthogonal to $H^-_J = \mathrm{span}\{\beta, J\beta\}$. This is written as

$$\omega' = \omega + u\,\beta + v\,J\beta\ , \tag{22}$$

where u, v are C^∞-functions. They satisfy

$$\int_M u|\beta|^2\, d\mu_g = \int_M v|\beta|^2\, d\mu_g = 0\ ,$$

and a differential equation corresponding to $d\omega' = 0$. Thus, any self-dual harmonic form is of type

$$c\,\omega' + a\,\beta + b\,J\beta,$$

where a,b,c are constants. The only condition for this form to be in $H^-_{J_{f,l,s}}$ is to be point-wise orthogonal to $\omega_{f,l,s}$ (see (21)). This amounts to

$$2cf' + al' + bs' = 0,$$

where

$$l' = l|\beta|^2, \quad s' = s|\beta|^2 \quad \text{and } f' = 2f + ul' + vs'\ . \tag{23}$$

Therefore we have the following statement.

Proposition 3.21. *Suppose (M, J) is a Kähler surface with holomorphically trivial canonical bundle. Let β be a closed form trivializing the canonical bundle. Consider a conformal class compatible with J and let g be the Gauduchon metric in this class. Let ω be the associated form and let $J_{f,l,s}$ be the g-related almost complex structure defined via (21). Then*

$$h^-_{J_{f,l,s}} = 3 - rank(\mathrm{Span}(f', l', s')),$$

with f', l', s' as in (23). The case $h^-_{J_{f,l,s}} = 0$ is the generic situation, thus the set of almost complex structures $\tilde{J}$ with $h^-_{\tilde{J}} = 0$ is dense in $\mathcal{J}_J$. The cases $h^-_{J_{f,l,s}} = 2$, $h^-_{J_{f,l,s}} = 1$ also occur.

Note that g is a hyperKähler metric precisely when $|\beta|^2 = 2$ pointwise and in this case $\omega' = \omega$.

Remark 3.22. We leave to the interested reader to check the computation that $h^-_{J_{f,l,s}} = 2$ if and only if

$$f = \pm(1 - k_1 u - k_2 v)|\beta| w, \ \ l = \pm 2k_1 |\beta|^{-1} w, \ \ s = \pm 2k_2 |\beta|^{-1} w,$$

where k_1, k_2 are arbitrary constants, u, v are given by (22), and

$$w = [(1 - k_1 u - k_2 v)^2 |\beta|^2 + 2(k_1^2 + k_2^2)]^{-1/2} \ .$$

We just observe that most of the examples with $h^-_{J_{f,l,s}} = 2$ described above are non-integrable almost complex structures. This can be again checked directly, or one can argue as follows. If for a certain metric g and functions f, l, s, we obtain an *integrable* almost complex structure $J_{f,l,s}$, then $(g, J, J_{f,l,s})$ is a bi-Hermitian structure. It is well known that conformal classes carrying bi-Hermitian structures are very particular, as Theorem 2 in [7] shows. On the other hand, our Proposition 3.21 shows that examples of almost complex structures $J_{f,l,s}$ with $h^-_{J_{f,l,s}} = 2$ occur in *each* conformal class associated to the given J. Thus, most of these $J_{f,l,s}$ must be non-integrable.

As an extension of Conjecture 2.5, it is natural to ask:

Question 3.23. *Are there (compact, 4-dimensional) examples of non-integrable almost complex structures J with $h^-_J \geq 2$ other than the ones arising from Proposition 3.21? In particular, are there any examples with $h^-_J \geq 3$?*

Theorem 1.1 follows from Theorem 3.12 and Propositions 3.19, 3.20, 3.21. □

3.5. **Applications of Theorem 1.1.** We end this section with a couple of applications of our main result. First, we prove that the C^∞-pure property no longer holds even for a Kähler J, if one gives up the compactness of the manifold.

Theorem 3.24. *Let (M, J) be a compact complex surface with non-trivial canonical bundle and non-zero geometric genus (equivalently, $h_J^- \neq 0$). Let B be a small contractible open set in M. Then the C^∞-full property for J still holds on $M \setminus B$, but the C^∞-pure property for J on $M \setminus B$ no longer holds.*

Proof. Since now is not obvious to which set the groups $H_J^\pm$ refer to, we'll use here the notations $H_J^\pm(M)$, $H_J^\pm(M \setminus B)$, etc. By Mayer-Vietoris, the inclusion $i : M \setminus B \hookrightarrow M$ induces an isomorphism in cohomology

$$H^2(M; \mathbb{R}) \xrightarrow{i^*} H^2(M \setminus B; \mathbb{R}) \ .$$

Via this isomorphism, the subgroups $H_J^\pm(M)$ inject in $H_J^\pm(M \setminus B)$, respectively. Thus, $(M \setminus B, J)$ still has the C^∞-full property.

For the C^∞-purity statement, let $\alpha \in \mathcal{Z}_J^-(M)$, $\alpha \not\equiv 0$. Choose a J-compatible metric g and a smooth function $r \geq 0$ compactly supported on B, so that $r^2|\alpha|_g^2 < 2$. Let $f = (1 - \frac{1}{2}r^2|\alpha|_g^2)^{1/2}$ and let $\tilde{J}$ be the almost complex structure defined by g and $\tilde{\omega} = f\omega + r\alpha$ as in (11). From Theorem 3.12, we have $H_{\tilde{J}}^-(M) = Span\{[J\alpha]\}$.

Consider now the cohomology class $[\alpha]$. By Theorem 2.2, $[\alpha] \in H_{\tilde{J}}^+(M)$. Thus, there exists a 1-form ρ on M such that $\alpha + d\rho$ is $\tilde{J}$-invariant. On the other hand, by construction, it is clear that $\tilde{J} = J$ on $M \setminus B$. Thus, on $M \setminus B$, $\alpha + d\rho$ is J-invariant, while α is obviously J-anti-invariant. Hence $i^*[\alpha] \in H_J^+(M \setminus B) \cap H_J^-(M \setminus B)$.

But the argument works for any $\alpha \in \mathcal{Z}_J^-(M)$. Thus, we get

$$i^*(H_J^-(M)) \subset H_J^+(M \setminus B) \quad \text{and} \quad i^*(H_J^+(M)) \subset H_J^+(M \setminus B) \ , \ \text{so}$$

$$i^*(H^2(M; \mathbb{R})) = H^2(M \setminus B; \mathbb{R}) = H_J^+(M \setminus B).$$

Therefore, we obtain

$$H_J^+(M \setminus B) \cap H_J^-(M \setminus B) = H_J^-(M \setminus B) \ ,$$

and the right hand-side is non-empty, as it contains at least $i^*(H_J^-(M))$. $\square$

Next, we show that our examples of non-integrable almost complex structures with $h_{\tilde{J}}^- = 2$ from Proposition 3.21 cannot admit a smooth pseudo-holomorphic blowup.

Theorem 3.25. *Suppose $\tilde{J}$ is a non-integrable almost complex structure with $h_{\tilde{J}}^- = 2$ on a $K3$ surface (or on T^4) and assume also that $\tilde{J}$ is metric related to a complex structure. Then there is no* smooth *almost complex structure $\tilde{J}'$ on $K3\#\overline{\mathbb{CP}^2}$ (or on $T^4\#\overline{\mathbb{CP}^2}$) so that the blowup map $f : K3\#\overline{\mathbb{CP}^2} \to K3$ (or $f : T^4\#\overline{\mathbb{CP}^2} \to T^4$) is a $(\tilde{J}', \tilde{J})$ holomorphic map. In other words, there is no pseudoholomorphic blowup for such a $\tilde{J}$.*

Proof. If there is such a $\tilde{J}'$, it should satisfy:

(1) $\tilde{J}'$ is not integrable;

(2) $h^-_{\tilde{J}'} = 2$;
(3) $\tilde{J}'$ is metric related to a complex structure.

However, by our Theorem 1.1, there are no such almost complex structures on $K3\#\overline{\mathbb{CP}^2}$ (or $T^4\#\overline{\mathbb{CP}^2}$). □

The above proposition should be compared with Usher's result [30]: there is always such a Lipschitz continuous almost complex structure J'. The same argument but with some modification of our previous definition can ensure that there is no such C^1 almost complex structure.

4. Well-balanced almost Hermitian 4-manifolds

In this section we introduce a class of 4-dimensional almost Hermitian structures that contains the Hermitian ones and the almost Kähler ones.

4.1. The image of the Nijenhuis tensor.

Given an almost complex structure J, at each point $p \in M$ define the image of its Nijenhuis tensor N_J by

$$Im(N_J)_p = \mathrm{Span}\{N_J(X,Y)\,|X,Y \in T_pM\}.$$

This is J-invariant, that is if $Z \in Im(N_J)_p$, then $JZ \in Im(N_J)_p$. The specific of dimension 4 is that at each point $Im(N_J)_p$ is either 0, or 2-dimensional, but never 4-dimensional. This is so, because N_J can be seen as a map

$$N_J : T_J^{2,0} \to T_J^{0,1}, \;\; N_J(Z_1 \wedge Z_2) = N_J(Z_1, Z_2) = [Z_1, Z_2]^{0,1}, \; Z_1, Z_2 \in T_J^{1,0},$$

and in dimension 4 the bundle $T_J^{2,0}$ is real 2-dimensional. Here the superscripts denote the usual complex type of vectors and forms induced by J.

One can ask when is $Im(N_J)$ a distribution over M. This certainly happens when J is integrable, as by Nirenberg-Newlander theorem this holds if and only if $N_J = 0$ everywhere. To ask that $Im(N_J)$ is everywhere 2-dimensional on M is equivalent to say that N_J is non-vanishing at each point. As the Nijenhuis tensor can be seen as a section of the bundle $\Lambda_J^{2,0} \otimes T_J^{0,1}$, John Armstrong observed ([8], Lemma 3) that the non-vanishing of N_J at each point has topological consequences.

Proposition 4.1. *([8]) If (M,J) is a 4-dimensional compact almost complex manifold with N_J non-vanishing at each point, then the signature and Euler characteristic of M satisfy*

$$5\chi(M) + 6\sigma(M) = 0 \,.$$

4.2. The well-balanced condition.

The following is a classical result (see, for instance, [20])

Proposition 4.2. *Let (M,g,J,ω) be an almost Hermitian manifold. Then*

$$(\nabla_X \omega)(\cdot,\cdot) = 2 < N_J(\cdot,\cdot), JX > + \frac{1}{2}\Big(d\omega(X,\cdot,\cdot) - d\omega(X,J\cdot,J\cdot)\Big) \tag{24}$$

It is well known that in dimension 4, there are just two Gray-Hervella [15] classes of special almost Hermitian manifolds – Hermitian and almost Kähler ones. These correspond to the vanishing (for any X) of the first, respectively second term on the right side of (24). In fact, on a general 4-dimensional almost Hermitian manifold, let θ be the Lee form defined be $d\omega = \theta \wedge \omega$. Then a short computation shows that

$$\frac{1}{2}\Big(d\omega(X,\cdot,\cdot) - d\omega(X, J\cdot, J\cdot)\Big) = ((JX)^\flat \wedge \theta)'', \tag{25}$$

where the superscript $''$ denotes the J-anti-invariant part of a 2-form. It is clear that the right hand-side of (25) vanishes for all X if and only if $\theta = 0$, i.e. $d\omega = 0$.

Relaxing both the Hermitian and the almost Kähler conditions, it is natural to ask that for every X at least one (but not necessarily the same) of the terms in the right hand-side of (24) vanishes. From the observations above, we know that in dimension 4 the Nijenhuis term vanishes for at least a two dimensional space at each point. The proof of the following proposition is tedious (but straightforward), so we just sketch it, leaving the interested reader to fill in remaining details.

Proposition 4.3. *Let (M^4, g, J, ω) be a 4-dimensional almost Hermitian manifold. Then the following statements are equivalent:*

(i) For any $p \in M$ and any $X \in T_pM$, at least one of the terms in the right side of (24) vanishes;

(ii) For any $p \in M$, $(N_J)_p = 0$, or $\theta_p^\sharp \in Im(N)_p$;

(iii) $\Big(\imath_{N_J(X,Y)} d\omega\Big)'' = 0$, for any $X, Y \in T_pM$ and $p \in M$;

(iv) For any local non-vanishing section $\psi \in \Omega_J^-$,

$$|\nabla\psi|^2 = |\nabla(J\psi)|^2, \;\; < \nabla\psi, \nabla(J\psi) >= 0.$$

Proof. Any (smooth) local section $\phi \in \Omega_J^-$ with $|\phi|^2 = 2$, determines (smooth) local 1-forms a, b, c by

$$\begin{aligned} \nabla\omega &= a \otimes \phi + b \otimes J\phi \\ \nabla\phi &= -a \otimes \omega + c \otimes J\phi \\ \nabla(J\phi) &= -b \otimes \omega - c \otimes \phi, \end{aligned} \tag{26}$$

We show that conditions (i), (ii), (iii), (iv) are all equivalent with:

(v) For any point $p \in M$, there exists an open set U containing p and a section $\phi \in \Omega_J^-$, defined on U, with $|\phi|^2 = 2$, so that the corresponding 1-forms a and b satisfy the pointwise conditions

$$|a|^2 = |b|^2 \text{ and } \;\; < a, b >= 0. \tag{27}$$

Note first of all that if the condition (27) is satisfied for a given section $\phi \in \Omega_J^-$ with $|\phi|^2 = 2$, then it holds for any other section $\tilde{\phi}$ with the same property (in other words, (27) is "gauge" independent). Indeed, let

$$\tilde{\phi} = \cos t\; \phi + \sin t\; J\phi,$$

for some smooth local function t. The corresponding 1-forms given by (26) change as

$$\begin{aligned} \tilde{a} &= a \cos t + b \sin t \\ \tilde{b} &= -a \sin t + b \cos t \\ \tilde{c} &= c + dt. \end{aligned} \tag{28}$$

Then it is easily checked that $\tilde{a}, \tilde{b}, \tilde{c}$ satisfy (27), assuming that a, b, c did so.

We prove now the equivalence (iv) $\Leftrightarrow$ (v). Given a section $\phi \in \Omega_J^-$, with $|\phi|^2 = 2$ and the 1-forms a, b, c defined by (26), one checks that

$$|\nabla\phi|^2 - |\nabla J\phi|^2 = 2(|a|^2 - |b|^2) \; , \quad < \nabla\phi, \nabla J\phi >= 2 < a, b > .$$

Hence, the implication (iv) $\Rightarrow$ (v) is clear. For the other implication, let $\psi \in \Omega_J^-$ be a local non-vanishing section and let $\phi = \frac{\sqrt{2}\psi}{|\psi|}$. Straightforward computations imply

$$|\nabla\phi|^2 - |\nabla J\phi|^2 = \frac{2(|\nabla\psi|^2 - |\nabla J\psi|^2)}{|\psi|^2} \; , \;\; < \nabla\phi, \nabla J\phi >= \frac{2 < \nabla\psi, \nabla J\psi >}{|\psi|^2},$$

and (v) $\Rightarrow$ (iv) follows now easily.

Using (25), the reader can check the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). We will show next that (i) $\Leftrightarrow$ (v). Let ϕ be a local section in Ω_J^-, with $|\phi|^2 = 2$. Then, using the symmetries of the Nijenhuis tensor and (25) one can check that

$$2 < N_J(\cdot, \cdot), JX >= m(X) \otimes \phi - Jm(X) \otimes J\phi,$$
$$\frac{1}{2}\Big(d\omega(X, \cdot, \cdot) - d\omega(X, J\cdot, J\cdot)\Big) = n(X) \otimes \phi + Jn(X) \otimes J\phi,$$

where m and n are local 1-forms. Thus, with respect to the chosen section ϕ, the 1-forms a, b given by (26) are given by

$$a = m + n \; , \;\; b = -Jm + Jn.$$

Easy computation shows that (27) is equivalent to

$$< m, n >= 0 \; , \;\; < m, Jn >= 0,$$

which is easily seen to be equivalent to (i).

□

Definition 4.4. *(i) An almost Hermitian manifold (M^4, g, J, ω) is called well-balanced if it satisfies one (and hence all) of the conditions in Proposition 4.3.*

(ii) An almost complex structure J on a 4-manifold M^4 is called well-balanced if it admits a compatible well-balanced almost Hermitian structure.

It is interesting to understand how large is the class of well-balanced almost complex structures on compact 4-manifolds, but we leave this problem for future study. Locally, any almost complex structure in dimension 4 is compatible with some symplectic form [22], so locally any almost complex structure is well-balanced.

The following result provides examples of well-balanced almost Hermitian 4-manifolds.

Proposition 4.5. *(i) Any 4-dimensional Hermitian or almost Kähler manifold is well-balanced.*

(ii) Suppose g is a Riemannian metric adapted to a complex-symplectic on a 4-manifold; in other words, assume that g is compatible to a triple I, J, K of almost complex structures satisfying the quaternion relations and assume that I is integrable, and that (g, J) and (g, K) are almost Kähler. Then for any constant angles t and s, the almost Hermitian structure $(g, \tilde{J})$ with $\tilde{J} = \cos t\, I + \sin t\, (\cos s\, J + \sin s\, K)$ is well-balanced.

(iii) Let M be a compact quotient by a discrete subgroup of the 3-step nilpotent Lie group G, whose nilpotent Lie algebra $\mathfrak{g}$ has structure equations

$$de^1 = de^2 = 0, de^3 = e^1 \wedge e^4, de^4 = e^1 \wedge e^2.$$

Consider the invariant metric $g = \sum(e^i \otimes e^i)$ and the compatible almost complex structure J given by $Je^1 = e^2$, $Je^3 = e^4$. Then (g, J) is well-balanced.

Proof. (i) In either case, it is obvious that condition (iii) of Proposition 4.3 is satisfied.

(ii) It is clear that it is enough to check the case $s = 0$. Let us denote $\omega_I, \omega_J, \omega_K$ the three fundamental forms. Since (g, I) is Hermitian and $(g, J), (g, K)$ are almost Kähler, we have

$$\begin{aligned} \nabla\omega_I &= a \otimes \omega_J + Ia \otimes \omega_K \\ \nabla\omega_J &= -a \otimes \omega_I - Ja \otimes \omega_K \\ \nabla\omega_K &= -Ia \otimes \omega_I + Ja \otimes \omega_J, \end{aligned} \tag{29}$$

for a 1-form a. Let $\tilde{\omega}$ the form corresponding to $\tilde{J} = \cos tI + \sin tJ$. Taking $\tilde{\phi} = \omega_K$, a short computation shows that

$$\nabla\tilde{\omega} = (Ia \cos t - Ja \sin t) \otimes \tilde{\phi} - a \otimes \tilde{J}\tilde{\phi},$$

and the statement is easily verified.

(iii) Direct computation shows that at each point $Im(N_J) = \mathrm{Span}(e_3, e_4)$. Even without computation, one can verify this by noting that the commutator $[\mathfrak{g}, \mathfrak{g}]$ is $\mathrm{Span}(e_3, e_4)$ and this is J-invariant, by the definition of J. Next, using the structure equations, one checks that $d\omega = -e^3 \wedge \omega$, where $\omega = e^1 \wedge e^2 + e^3 \wedge e^4$. Thus, condition (ii) of Proposition 4.3 is satisfied. □

Remark 4.6. Note that J from example (iii) in Proposition 4.5 is not integrable and cannot be tamed by a symplectic form on M (see Proposition 6.1 in [12] and Remark 3.2 above).

To state the main result of this subsection we need one more definition.

Definition 4.7. *An almost Hermitian manifold (M^4, g, J) has Hermitian type Weyl tensor if*

$$< W^+(J\beta), J\beta > = < W^+(\beta), \beta >, \text{ for any } \beta \in \Omega^-_J. \tag{30}$$

It is well known that if J is integrable (i.e. (M^4, g, J) is a Hermitian manifold), then (30) holds. Also, any almost Hermitian structure with an ASD metric trivially satisfies (30).

Theorem 4.8. *Let (M^4, J) be a compact almost complex 4-manifold which admits a compatible Riemannian metric g so that (g, J) is well-balanced and has Hermitian type Weyl tensor. Then $h_J^- = 0$ or J is integrable.*

Proof. Suppose β is a non-trivial closed, J-anti-invariant form on M. The next Lemma shows that, under the given assumptions, $J\beta$ is also closed, thus $\Phi = \beta + iJ\beta$ is a closed, complex form of (2,0) type. The integrability of J then follows (see e.g. [27]).

Lemma 4.9. *Let (M^4, g, J, ω) be a compact, almost Hermitian 4-manifold which is well-balanced and has Hermitian type Weyl tensor. Then for any $\beta \in \Omega_J^-$, $d\beta = 0 \Leftrightarrow d(J\beta) = 0$.*

Proof of Lemma: It's enough to prove $d\beta = 0 \Rightarrow d(J\beta) = 0$. The well-known Weitzenböck formula for a 2-form ψ is

$$\int_M (|d\psi|^2 + |\delta\psi|^2 - |\nabla\psi|^2)\, dV = \int_M (\frac{s}{3}|\psi|^2 - < W(\psi), \psi >)\, dV.$$

Applying this for β and $J\beta$ and using the assumption on the Weyl tensor, we get

$$\int_M (|d\beta|^2 + |\delta\beta|^2 - |\nabla\beta|^2)\, dV = \int_M (|d(J\beta)|^2 + |\delta(J\beta)|^2 - |\nabla(J\beta)|^2)\, dV.$$

Now, by assumption $\beta \in \Omega_J^-$ and $d\beta = 0$, thus β is harmonic, so it is non-vanishing on an open dense set in M. From the well-balanced assumption and continuity, we get that $|\nabla(J\beta)|^2 = |\nabla\beta|^2$ everywhere on M. Thus,

$$0 = \int_M (|d\beta|^2 + |\delta\beta|^2)\, dV = \int_M (|d(J\beta)|^2 + |\delta(J\beta)|^2)\, dV.$$

The lemma and the Theorem are thus proved. □

The following is an immediate consequence.

Corollary 4.10. *A compact 4-dimensional almost Kähler structure (g, J, ω) with Hermitian type Weyl tensor and with $h_J^- \neq 0$ must be Kähler.*

Remark 4.11. Under different additional conditions, some other integrability results have been obtained for compact, 4-dimensional almost Kähler manifolds (g, J, ω) with Hermitian type Weyl tensor (see [4], [5]).

Remark 4.12. The corollary implies that if we start with a Kähler surface (M, g, J, ω) and define the almost complex structures $\tilde{J}_\alpha^\pm$ corresponding to (11) and (14) for $\alpha \in \mathcal{Z}_J^-$, then $\tilde{J}_\alpha^\pm$ cannot admit compatible almost Kähler structures with Hermitian-type Weyl tensor. They do admit compatible almost Kähler structures (since $\pm\omega + \alpha$ is symplectic).

5. Symplectic Calabi-Yau equation and semi-continuity property of $h_J^\pm$

In this section, we use the beautiful ideas in [10] to establish a stronger semi-continuity property for $h_J^\pm$ than in Theorem 2.6, near an almost complex structure which admits a compatible symplectic form.

5.1. Symplectic CY equation and openness.

The classical Calabi-Yau theorem can be stated as follows: Let $(M, J, \tilde\omega)$ be a Kähler manifold. For any volume form σ satisfying $\int_M \sigma = \int_M \tilde\omega^n$, there exists a unique Kähler form ω with $[\omega] = [\tilde\omega]$ s.t. $\omega^n = \sigma$.

Yau's original proof of the existence ([32]) makes use of a continuity method between the prescribed volume form σ and the natural volume form $\tilde\omega^n$. The proof of openness is by the implicit function theorem. The closedness part is obtained by a priori estimates.

5.1.1. *Set up.* In [10], Donaldson introduced the symplectic version of the Calabi-Yau equation.

Let (M, J) be a compact almost complex $2n-$manifold and assume that Ω is a symplectic form compatible with J. For any function F with

$$\int_M e^F \tilde\omega^n = \int_M \tilde\omega^n \tag{31}$$

the symplectic CY equation is the following equation of a $J-$compatible symplectic form $\tilde\omega$,

$$\omega^n = e^F \tilde\omega^n. \tag{32}$$

In [10], Donaldson further observed that solvability of the symplectic CY equation in dimension 4 may lead to some amazing results in four dimensional symplectic geometry.

5.1.2. *Openness.* In Donaldson's paper, he proves that the solution set of the symplectic CY equation (32) is open by using the implicit function theorem. This only works for dimension 4. Donaldson actually works in the general setting of $2-$forms on 4 manifolds.

Suppose M is a $4-$manifold with a volume form ρ and a choice of almost-complex structure J. At any point x, ρ and J induce a volume form and a complex structure on the vector space $T_x(M)$. Denote by P_x the set of positive $(1,1)-$forms whose square is the given volume form. Then P_x is a three-dimensional submanifold in $\Lambda^2 T_x(M)$ (a sphere in a $(3,1)-$space). We consider the $7-$dimensional manifold $\mathcal{P}$ fibred over M with fiber P_x,

$$\mathcal{P} = \{\omega^2 = \rho \,|\, \omega \text{ is compatible with } J\}.$$

It is a submanifold of the total space of the bundle Λ^2.

Now, we want to find a symplectic form ω which is compatible with J and has fixed volume form with some cohomology conditions. That is, we

are searching for ω satisfying the following conditions (we call this condition type D):

$$\begin{cases} \omega & \subset \mathcal{P}_\rho, \\ d\omega & = 0, \\ [\omega] & \in e + H^2_+ \subset H^2(M;\mathbb{R}). \end{cases} \tag{33}$$

Here e is a fixed cohomology class and H^2_+ is a maximal positive subspace. Notice, we have three families of variables: ρ, J and e. In particular, e varies in a finite dimensional space.

We have the following result which is a slight variation of Proposition 1 in [10].

Proposition 5.1. *Suppose ω is a solution of type D constrain with given $\mathcal{P}$ and e. If we have a smooth family $\mathcal{P}^{(b)}$ parameterized by a Banach space B, $\{\mathcal{P}^{(b)}\}$, with $\mathcal{P} = \mathcal{P}^{(0)}$ and b varies in B, then we have a unique solution of the deformed constraint in a sufficiently small neighborhood of 0 in B. Further, this solution lies in a small C^0 neighborhood of ω.*

We just indicate how to find a small neighborhood for which we have the existence.

For each point $x \in M$, the tangent space to P_x at $\omega(x)$ is a maximal negative space. Thus the solution ω determines a conformal structure on M. We fix a Riemannian metric g in this conformal class (actually, we can choose the metric determined by ω and J). For small η, $\omega + \eta$ lying in $\mathcal{P}_\rho$ is expressed as

$$\eta^+ = Q(\eta),$$

where Q is a smooth map with $Q(\eta) = O(\eta^2)$. After choosing $2-$form representatives of H^2_+, closed forms $\omega + \eta$ satisfying our cohomological constraint can be expressed as $\omega + da + h$ where $h \in H^2_+$ and where a is a $1-$form satisfying the gauge fixing constraint $d^*a = 0$. Thus our constraints correspond to the solutions of the PDE

$$\begin{cases} d^*a & = 0 \\ d^+a & = Q(da + h) - h^+. \end{cases} \tag{34}$$

Thus, our constraints are represented by a system of nonlinear elliptic PDE. Donaldson further observes that its linearization

$$L = d^* \oplus d^+ : \Omega^1/\mathcal{H}^1 \longrightarrow \Omega^0/\mathcal{H}^0 \oplus \Omega^2_+/H^2_+$$

is invertible. Then we apply the following version of the implicit function theorem:

Theorem 5.2. *Let X, Y, Z be Banach spaces and $f : X \times Y \longrightarrow Z$ a Fréchet differentiable map. If $(x_0, y_0) \in X \times Y$, $f(x_0, y_0) = 0$, and $y \mapsto D_2 f(x_0, y_0)(0, y)$ is a Banach space isomorphism from Y onto Z. Then there exist neighborhoods U of x_0 and V of y_0 and a Fréchet differentiable function such that $f(x, g(x)) = 0$ and $f(x, y) = 0$ if and only if $y = g(x)$, for all $(x, y) \in X \times Y$.*

To use this theorem, first notice that $D_2 f$ is just our L defined above, which is invertible at a solution of our constraints. Moreover, X is our parametrization space B, Y is $(\Omega^1)_1/\mathcal{H}^1$, Z is $(\Omega^0)_0/\mathcal{H}^0 \oplus (\Omega^2_+)_0/H^2_+$. Here $(\Omega^n)_m$ represents the space of C^m $n-$forms.

Then every condition is satisfied in our setting.

5.2. **Semi-continuity properties of $h^\pm_J$.**

5.2.1. *Weak and strong neighborhoods.* As described in 3.1.1, the space of C^∞ almost complex structures $\mathcal{J} = \mathcal{J}^\infty$ is not a Banach manifold but a Fréchet manifold. In this case we can still apply Proposition 5.1 to a smooth path or a finite dimensional space (hence Banach) in $\mathcal{J}$. That is to say, if an almost complex structure J has a solution of the CY equation $\omega^2 = \rho$ with a $J-$compatible form ω satisfying $[\omega] \in e + H^2_+$, then for any path through J, there is a small interval near J such that the CY type equation is solvable with conditions in (D_t) in this interval. In the end we get a weak neighborhood–the union of all the intervals. Notice that this is not necessarily "a small ball" near J, i.e. it may not have an interior point.

We would like to apply Proposition 5.2 to an open neighborhood with respect to the C^∞ topology, which can be called a strong neighborhood compared with the one described above. For this purpose, notice that the tangent space $T_J\mathcal{J}^l$ at J consists of C^l-sections A of the bundle $End(TM, J)$ such that $AJ + JA = 0$. It is a Banach space with C^l norm. Moreover, this gives rise to a local model for $\mathcal{J}^l$ via $Y \longmapsto J\exp(-\mathrm{JY})$. Thus we can apply Proposition 5.1 to a Banach chart of J in the space of C^l almost complex structures $\mathcal{J}^l$ endowed with C^l norm.

Corollary 5.3. *If we parameterize $\mathcal{P}^{(b)}$ in Proposition 5.1 by a neighborhood $U(J_0)$ of J_0 in $\mathcal{J}$ with C^∞ topology, then we can get a small neighborhood of J satisfying all the properties stated in Proposition 5.1 under the same topology.*

Proof. The space of C^1 almost complex structures $\mathcal{J}^1$ with C^1 norm is a Banach manifold. We parameterize a neighborhood of J_0 by an open set in the induced Banach space.

Then we can apply Proposition 5.1 for this setting. The only point we need to check is that the Fréchet differentiability of the reliance of our constraints with respect to the parametrization space $\mathcal{J}^1$. Here, we adapt the arguments in [31]. We define a tensor Π (which is denoted by $\mathcal{P}$ in [31]) as

$$\Pi^{ij}_{kl} = \frac{1}{2}(\delta^i_k\delta^j_l - J^i_k J^j_l).$$

When restricting Π on the space of $2-$forms, it is just the projection onto the $J-$anti-invariant part. As in [31], we also define $\chi_1, \cdots, \chi_r$ be self-dual harmonic $2-$forms with respect to ω such that $\{\omega, \chi_1, \cdots, \chi_r\}$ are L^2 orthogonal bases for $\mathcal{H}^+_\omega$.

Consider the operator $\Phi : (\Omega^1)_1 \times \mathbb{R}^r \times \mathcal{J}^1 \longrightarrow (\Omega^2)_0$ by

$$\Phi(b,\underline{s},J)=(\log\frac{(\omega+\sum_{i=1}^r s_i\chi_i+db)^2}{\omega^2})\frac{(Id-\Pi_J)\omega}{2}+\Pi_J(\omega+\sum_{i=1}^r s_i\chi_i+db),$$

The solution of $\Phi(b,\underline{s},J)=0$ gives a closed, $J-$invariant form with the same volume form as ω. In other words, we get a description of our constraints by the zero set of a map. It is easy to see that the map Φ is a Fréchet differentiable map.

Thus by Proposition 5.1 we have a neighborhood $U^1(J_0)$ of J_0 in which we have all the properties stated there. Especially, we can suppose $U^1(J_0)$ is a ball in $\mathcal{J}$ with radius ϵ.

Finally, the small neighborhood of J_0 in C^∞ with $d(J_0,J)<\frac{\epsilon}{2(1+\epsilon)}$, where d is defined in (9), is what we want. □

5.2.2. *Variations of $h_J^\pm$.* Following [24], given a compact almost complex manifold (M,J) define the *J-compatible symplectic cone*

$$\mathcal{K}_J^c=\left\{[\omega]\in H^2_{dR}(M;\mathbb{R})\ \mid \omega \text{ symplectic and } J \text{ is } \omega\text{-compatible}\right\}.$$

It is easy to see that $\mathcal{K}_J^c$ is an open convex set in H_J^+. It is also immediate from the definition that $\mathcal{K}_J^c\neq\emptyset$ if and only if J admits compatible symplectic forms.

Theorem 5.4. *Suppose M is a $4-$manifold with an almost complex structure J such that $\mathcal{K}_J^c(M)\neq\emptyset$. is non-empty. Then for any almost complex structure J' in a sufficiently small neighborhood of J as in Corollary 5.3, we have*

- $\mathcal{K}_{J'}^c(M)\neq\emptyset$;
- $h_J^+(M)\leq h_{J'}^+(M)$;
- $h_J^-(M)\geq h_{J'}^-(M)$.

Proof. The first statement is a direct consequence of Corollary 5.3, and was already observed by Donaldson (see also [22]).

As $\mathcal{K}_{J'}^c(M)$ and $\mathcal{K}_J^c(M)$ are nonempty open sets in $H_{J'}^+(M)$ and $H_J^+(M)$ respectively, to estimate $h_J^+(M)$ and $h_{J'}^+(M)$, we only need to estimate the dimensions of $\mathcal{K}_{J'}^c(M)$ and $\mathcal{K}_J^c(M)$.

Let $h=h_J^+(M)$. We choose h rays which are "in general position", i.e. the interior of their span is an open set of $\mathcal{K}_J^c(M)$. We suppose the h rays are $C\cdot[\omega_i]$'s where ω_i's are the $J-$compatible forms and $[\omega_i]$'s have homology norm 1 with respect to some bases.

Then we use Corollary 5.3 for each i with fixed volume form ω_i^2. Then we have h neighborhoods U_i such that for $J'\in U_i$, we have a J' compatible form ω_i' which is a small perturbation of ω_i. Let U be the intersection of these h neighborhoods. Then for any $J'\in U$, we have ω_i''s which are still in the general position (because they are perturbed in C_0 norm from a general position). And we see that the span of the h new rays belongs to $\mathcal{K}_{J'}^c(M)$ because positive combinations of ω_i''s are still $J'-$compatible forms. Hence

we have $h_J^+(M) \leq h_{J'}^+(M)$. The last inequality is a consequence of the previous one and Theorem 2.2. $\square$

Remark 5.5. The first statement also means that, on a $4-$manifold, the space of almost Kähler complex structure $\mathcal{J}_{ak}$ is an open subset of $\mathcal{J}$. If one considers complex deformation, the analogue of the first statement is a classical theorem of Kodaira and Spencer. Their theorem is in fact valid for any even dimension.

Let us consider the stratification

$$\mathcal{J} = \bigsqcup_{i=0}^{b^+} \mathcal{J}_i,$$

where $J \in \mathcal{J}_i$ if $h_J^- = i$. Then we have

Corollary 5.6. *On a $4-$manifold, $\mathcal{J}_0 \cap \mathcal{J}_{ak}$ is open in $\mathcal{J}$.*

It is known that $\mathcal{J}_{ak}$ is never the full space $\mathcal{J}$. In fact, in any connected component of $\mathcal{J}$ there are non-tamed almost complex structures (see e.g. [10]). Nonetheless Corollary 5.6 is a strong evidence of Conjecture 2.4. In addition, the path-wise semi-continuity established in Theorem 2.6 indicates that the strong semi-continuity property of Theorem 5.4 very likely holds for every J. This would imply that $\mathcal{J}_0$ is open in $\mathcal{J}$.

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